Symmetrical Vortex Knots

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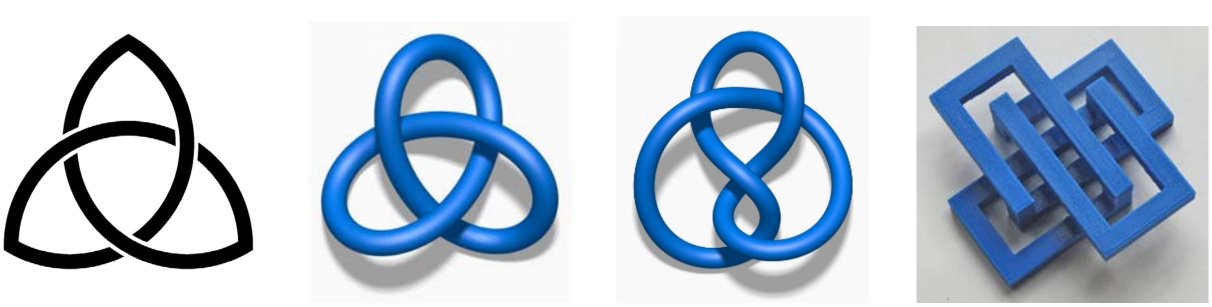
Abstract

My goal is to create sculptural models of mathematical knots that exhibit a high degree of symmetry. I start with an arbitrary torus knot and turn it into a circular braid with the same symmetry by selectively reversing some of the displayed crossings in a knot projection along the rotational symmetry axis. In the simplest case, I make the knot completely alternating. The resulting braid can then be draped around a cylinder to result in a classical Turk's Head Knot. Alternatively, the braid can be fit around a sphere to yield a ball-shaped knot. Different choices for the sequence of over- and under-crossings result in different mathematical knots. The approach can be generalized by choosing an appropriate short braid segment and stitching together a number of identical copies of it. The resulting knots all have the common property that the knot filament travels around the rotational axis in a monotonic manner; I call such knots "Vortex Knots." Small scale models, about five inches in diameter, have been made by additive manufacturing on a Fused Deposition Modeling (FDM) machine. To make a truly artful knot sculpture, a more interesting cross-section, e.g., a crescent shape, is swept along the knot curve, and the plastic maquette is turned into a small bronze sculpture, perhaps embellished with a colorful patina.

Possible Knot Symmetries

The type of symmetry that mathematical prime knots can exhibit is rather limited [3]. Except for the trivial *Unknot*, they cannot display simple mirror symmetry [6]. Their symmetries are limited to just three families of rotational symmetry groups:

- "**C*n***" (Schönfliess notation [10]) or "***nn***" (Conway's Orbifold notation [9]) exhibits one *n*-fold rotational symmetry axis (Figure 1a); here the axis is perpendicular to the plane of the paper.
- "**D*n***" or "***nnn***" has the same type of rotation axis, but also has *n* 2-fold rotation axes perpendicular to the primary *n*-fold axis (Figure 1b). In the simplest case, $\mathbf{D_2}$, there is just a single 2-fold rotation axis (Figure 1c); here it lies in the plane of the paper.
- "$\mathbf{S_{2n}}$" or "***n***X" also has a primary *n*-fold rotation axis, and in addition exhibits glide symmetry, involving a reflection along the primary axis combined with a rotation through an angle of 360°/*n* around that axis (Figure 1d).



(a) (b) (c) (d)

Figure 1: *Trefoil-knot (Knot 3_1): (a) 2D diagram with C_3 symmetry; (b) 3D model with D_3 symmetry. Figure-8 knot (Knot 4_1): (c) 3D model with C_2 symmetry; (d) 3D model with S_4 symmetry.*

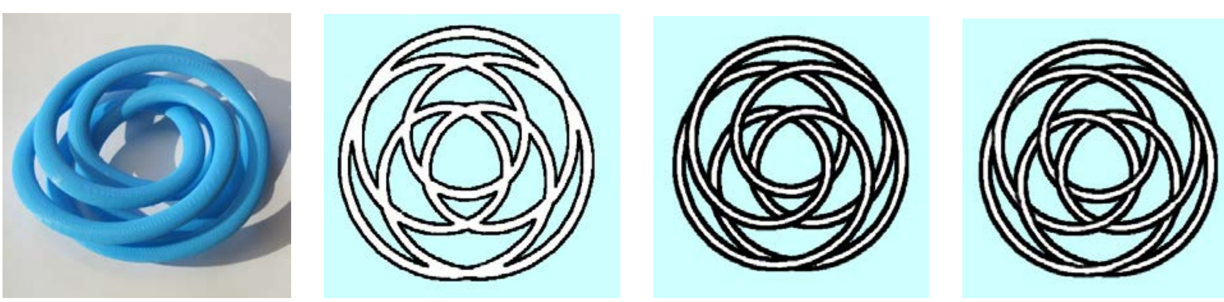
Most prime knots of low complexity can be re-shaped to display more geometrical symmetry than is implied by the depictions in the Rolfsen Knot Tables [5]. All but one knot (Knot 8_{17}) [4] that have eight or less crossings, can exhibit one of the above rotational symmetries. However, there seems to be no robust algorithm that automatically finds the maximal symmetry of a given particular knot. In order to find a possible symmetrical configuration of a given knot, one must use some ad-hoc trial-and-error approach. One might start with a projection of the given knot and apply various *Reidemeister-moves* [6] to shift some trace segments in the knot projection across one another without changing the topology of the knot. However, it is not clear what sequence of moves should be applied to tease out a projection that displays a higher degree of symmetry. Alternatively, one might form the knot of interest from a loop of wire or with some pipe-cleaners or chenille stems. Yet again, there is no recipe for the sequence of deformations that might bring about a more symmetrical configuration.

In view of this, a more effective way of creating mathematical knots of high symmetry is to use a procedural approach to define a path that moves through Euclidean 3-space, R^3, following a symmetrical pattern. In the following I outline a few techniques for generating such symmetrical paths. I might not know the label of the generated knot in the knot table, but the result is still useful to produce a knot sculpture with high symmetry. A tool like *SnapPy* [2] can help identify the generated knot, if its crossing number is not too high.

From Torus Knots to Vortex Knots

The most obvious and quick way to obtain a prime knot with *s*-fold rotational symmetry is to construct a TorusKnot(*s*,*t*), where the knot filament shoots through the central hole of a donut *s* times, while completing *t* turns around the hole (Figure 2a). These knots exhibit D_s symmetry. But, as the values of *s* and *t* get larger, the result does not look so much like a complex "Gordian" knot, but more like the cooling pipes in a power plant.

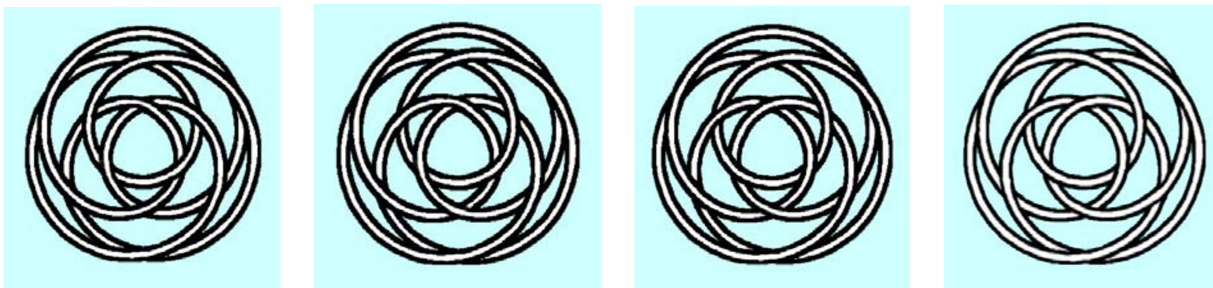
However, this can be remedied by turning the torus knot into a more intricate circular braid. First, the torus knot is squashed into a 2D circular template by setting all *z*-values to zero (Figure 2b). Now there is a choice how one might alter the original sequence (oooo uuuu)^3 of the over- and under-passes at all crossing points (Figure 2c). In order to maintain a symmetrical configuration, one must respect the original *s*-fold symmetry of the initial torus knot, or, at the very least, choose the crossing pattern to follow an integral subset, *s/I*, of that symmetry group. A first choice that immediately presents itself is to turn the flat 2D template into a strictly alternating knot (Figure 2d), resulting in a Turk's Head Knot [11]. The original TorusKnot(3,5) (Figure 2c) corresponds to Knot 10_{124} in the knot table, while the alternating knot (Figure 2d) is Knot 12a1019.



(a) (b) (c) (d)

Figure 2: *(a) TorusKnot(3,5); (b) flattened template; (c) braid representing the original knot; (d) an alternating over- under-crossing pattern, resulting in a Turk's Head Knot.*

Of course, the sequence of over- and under-passes can be made more intricate. For the above example, starting with a TorusKnot(3,5), there are a total of six different ways of choosing the over-/under-sequence so that one obtains different knots while maintaining full D_3 symmetry. The remaining four possibilities are shown in Figure 3. Figure 3a uses the pattern (ooou ouuu)^3; this yields Knot 11_{387}. The pattern (oouu oouu)^3 (Figure 3b) results in Knot 12n708. Similarly, pattern (oouo uouu)^3 (Figure 3c) results in Knot 12n839, and (ouuo uoou)^3 (Figure 3d) produces Knot 12n837.

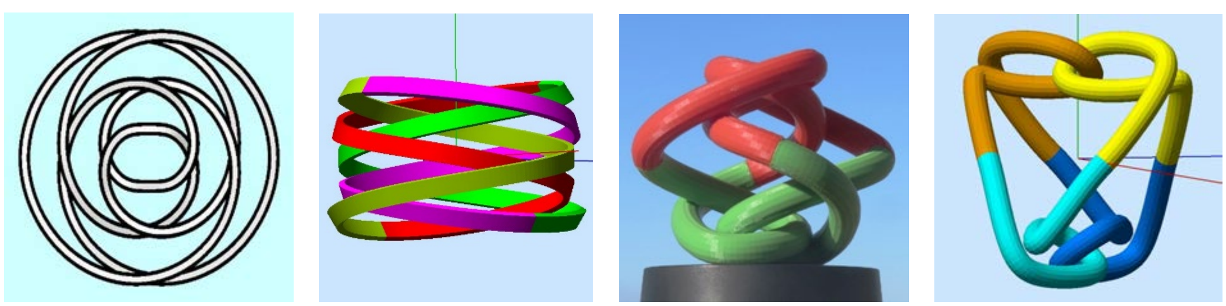


(a) (b) (c) (d)

Figure 3: *Circular braids: (a) (ooou ouuu)^3; (b) (oouu oouu)^3; (c) (oouo uouu)^3; (d) (ouuo uoou)^3.*

These circular braids still look rather flat. To obtain attractive 3D sculptures, I need to give these knots more "3-dimensionality." Figure 4 shows a few ways how this can be done. Here I started with TorusKnot(2,5) and changed the crossing pattern to the sequence (oouu oouu)^2 for subsequent over- and under-passes (Figure 4a); this produces Knot 6_3 in the knot tables. This flat, circular braid is now rotated through itself like a smoke ring. A 90° rotation results in a cylindrical braid (Figure 4b). I can drape this braid around a sphere or around an ellipsoid, rather than around a cylinder, to obtain a more ball-shaped sculpture (Figure 4c). Different colors in the upper half and the lower half of this sculpture, make it easier to see that the two halves can be transformed into one another through a 90° rotation around the *z*-axis with a simultaneous mirroring along the *z*-axis; this is the hallmark of S_4 symmetry.

This particular symmetry can be made even more visible, if the top- and bottom-pair of inter-linked lobes are pulled apart so that they form stretched, open chain links. These links can be seen as some kind of double-covering on two of the six edges of the tetrahedral frame. The top-pair and the bottom-pair are twisted in opposite directions, reflecting the mirroring operation inherent in the S_4 glide symmetry. During all these deformations, the knot topology remains unchanged; it happens to correspond to Knot 6_3 in the knot table. These transformations also preserve the property of a vortex knot that the filament spirals around the rotation axis in a monotone way.



(a) (b) (c) (d)

Figure 4: *Knot 6_3 shown: (a) as a flat 5-strand circular braid, (b) as a cylindrical braid, (c) as a ball-knot, (d) in a tetrahedral configuration with linked top- and bottom-edges.*

Figures 5 and 6 show additional examples of alternating Turk's Head Knots, THK(*s*,*t*), that were turned into more spherical BallKnots(*s*,*t*). The symmetry parameter *s* defines the number of *bights* or "bays" in the Turk's Head Knot, and the parameter *t* corresponds to the number of *leads* or "parts" in that knot. Depending on whether *t* is even or odd, the overall *s*-fold rotational symmetry will be of type D_s or S_{2s}, respectively. In either case, the filament takes *t* turns around the z-axis.

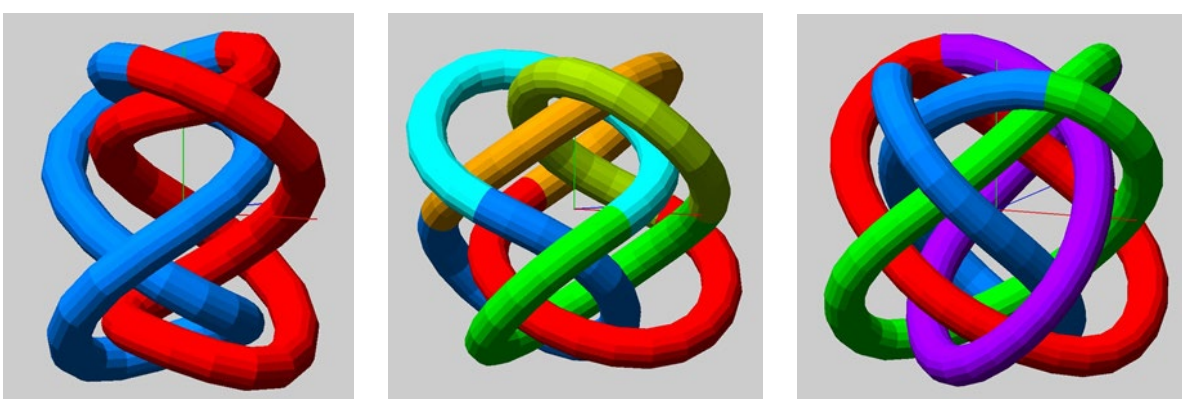
(a) (b) (c)

Figure 5: *BallKnots(s,t) based on Turk's Head Knots, THK(s,t), and their symmetries: (a) THK(4,3): S_8; (b) THK(3,4): D_3; (c) THK(4,5): S_8.*

(a) (b) (c)

Figure 6: *BallKnots(s,t) based on Turk's Head Knots, THK(s,t), and their symmetries: (a) THK(5,6): D_5; (b) THK(6,5): S_{12}; (c) THK(6,7): S_{12}.*

In Figure 7, I am using the pattern (oouu oouu)^s with 2-, 3-, and 4-fold rotational symmetry, and the knot strand is always circling the *z*-axis five times. The results are presented in the form of BallKnots(s,5). In Figures 7a and 7c, the differently colored parts are all identical and show the *s*-fold rotational symmetry. In Figure 7b, the knot has been split into upper and lower lobes, which are mirror images of one another.

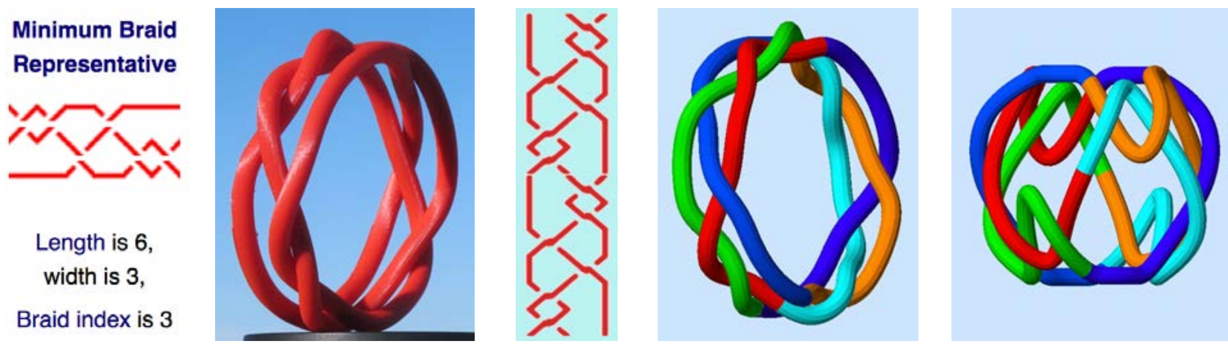


(a) (b) (c)

Figure 7: *BallKnots, BK(s,t), using crossover-pattern (oouu oouu)^s and their symmetries: (a) BK(2,5): S_4, (b) BK(3,5): S_6, (c) BK(4,5): S_8.*

Generalized Vortex Knots

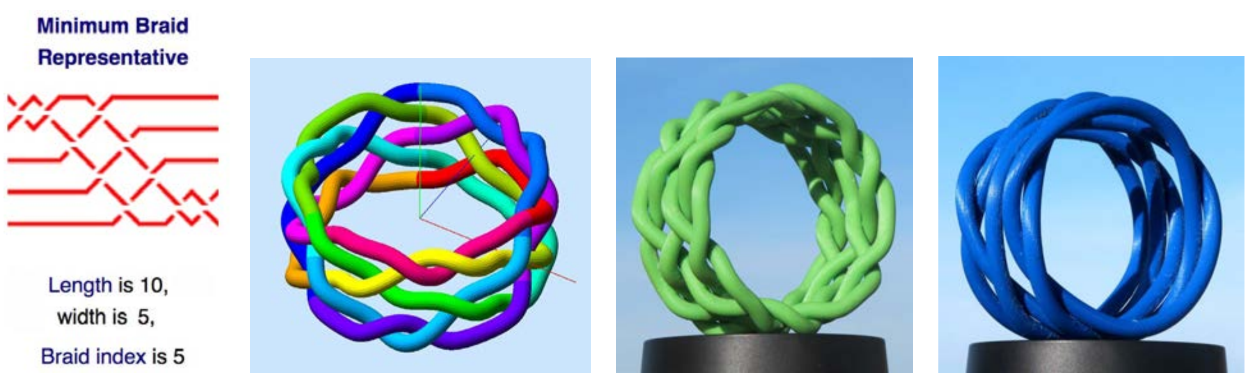
All the results so far, derived from some torus knots, have the property that the knot strand undulates s times in a regular manner back and forth from one edge of the braid to the other one. I will refer to the resulting knots as *regular* VortexKnots(*s*,*t*). Now I generalize the braid behavior so that in some regions the undulations of the strand have smaller amplitudes. One way to accomplish this is to start with a proper braid segment, e.g., the *minimum braid representation* of a Knot 6_3 (Figure 8a). This braid by itself does not exhibit a repetitive, periodic structure that would then automatically lead to a configuration with rotational symmetry when the braid is closed end-to-end to form a circular loop (Figure 8b). To remedy this situation, I concatenate *s* copies of such a braid into a longer braid, which then will yield a construction with *s*-fold rotational symmetry. However, one must be careful; not every value of *s* will lead to a single knot; some values will produce a link. For the 3-strand braid of Knot 6_3 (Figure 8a), two concatenated copies (Figure 8c) will result in a proper generalized vortex knot (Figures 8d, 8e); but concatenating three copies would result in complex link of three loops.



(a) (b) (c) (d) (e)

Figure 8: *(a) The braid representation of Knot 6_3, (b) Knot 6_3 as a cylindrical braid sculpture. (c) Two concatenated braids, (d) closed into a cylindrical braid, (e) formed into a ball-knot.*

For another example, I start with the 5-strand minimum braid representation of Knot 10_{43} (Figure 9a). Concatenating three of these braid segments into a cylindrical loop (Figure 9b) results in a generalized vortex sculpture with C_3 symmetry (Figure 9c). If I use only two copies of the braid segment, I obtain a sculpture with C_2 symmetry (Figure 9d).

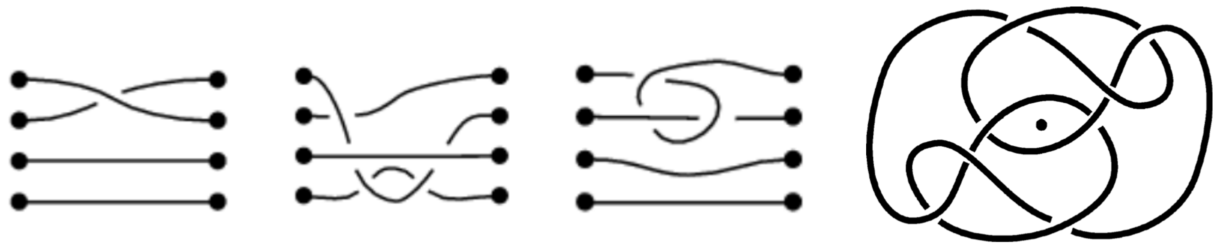


(a) (b) (c) (d)

Figure 9: *(a) The braid representation of Knot 10_{43}. (b) Three concatenated braids in a loop; (c) resulting sculpture with C_3 symmetry, (d) C_2 sculpture formed with 2 braid segments.*

Beyond Vortex Knots: "Braids" that Loop Back

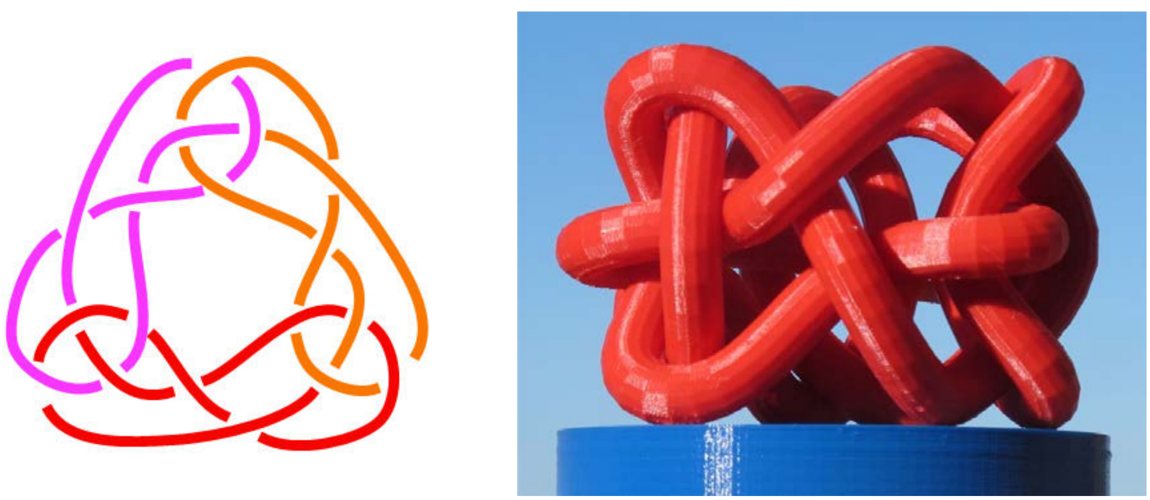
Proper braids (Figures 10a, 10b) must not have any backward turns as shown in Figure 10c. That latter braid, when closed into a loop, would no longer yield a proper vortex knot in which the knot filament travels in a monotone manner around the primary rotation axis. Figure 10d shows an example of a nice knot with 2-fold rotational symmetry; but this is no longer a vortex knot.



(a) (b) (c) (d)

Figure 10: *(a, b) Two proper representations of the same braid. (c) Not a true braid. (d) a symmetrical knot that is not a proper vortex knot.*

Figure 11a gives another example of a knot projection with C_3 symmetry that is not a vortex-knot; but it still makes a rather attractive 3D model, when this "bad" braid is wrapped around a cylinder (Figure 11b).



(a) (b)

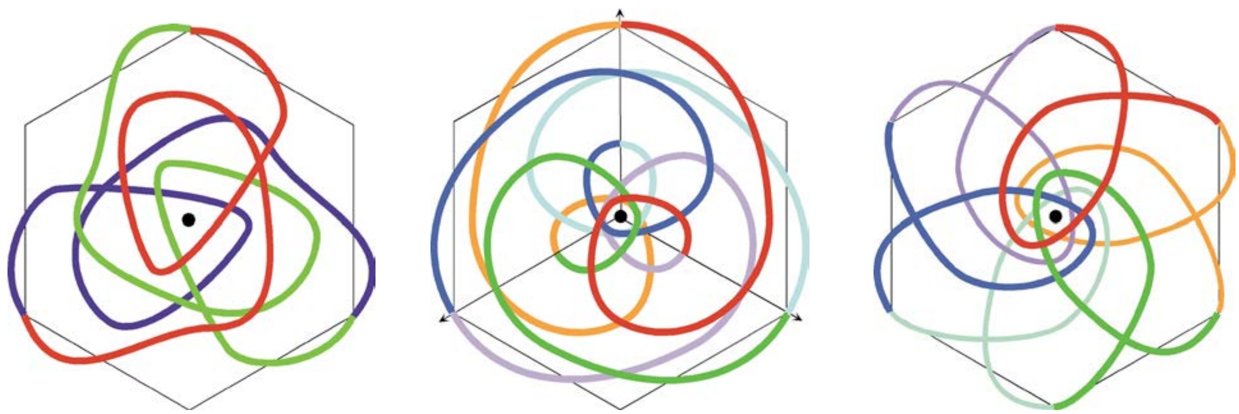
Figure 11: *(a) Another C_3-symmetrical knot; (b) its realization as a cylindrical braid.*

CAD Modeling

The various 3D models shown above have been constructed with Berkeley SLIDE [7], but any other CAD system would work just as well. The key step is to create an iterated process that creates the desired knot curve in 3D, which can then be used as the sweep path for a "fleshed-out" physical knot model. In most of the above models, I have defined a sequence of control points for a cubic B-spline. Typically, I use four times as many control points as there are crossings in the knot projection. I place two control points at every crossing, and the separation between them allows me to control the clearance between the crossing knot strands. In addition, there is an additional control point between subsequent crossings, which I use to create fairly smooth transitions between the crossings. This is good enough for the simple plastic models used to demonstrate the knot symmetries. For any final artistic sculpture models, I would use some additional control points, or I would employ a quartic B-spline rather than a cubic one.

Summary and Discussion

Symmetrical prime knots all have a dominant ***n***-fold rotational symmetry axis, because their overall symmetry must fall into one of the three families $\mathbf{C}_n$, $\mathbf{D}_n$, or $\mathbf{S}_{2n}$ (Schönfliess notation [10]) or "***nn***", "***nnn***", or "***n***X" (Conway's Orbifold notation [9]). A knot with ***n***-fold rotational symmetry of type $\mathbf{C}_n$, thus can be constructed by grouping ***n*** copies of an appropriate tangle of trace segments symmetrically around a common rotation center (Figure 12a). The symmetry of the resulting construction can be further enhanced, and the number of group elements doubled, by adding copies of each tangle that are flipped through an angle of 180° around an axis that intersects the dominant rotation axis at right angle; this yields $\mathbf{D}_n$ symmetry (Figure 12b). Alternatively, the number of tangles can be doubled by interspersing into the first $\mathbf{C}_n$-set of tangles another set of ***n*** tangles that are mirrored along the dominant rotation axis (Figure 12c).



(a) (b) (c)

Figure 12: *Knot constructions with **n**-fold rotational symmetry (**n**=3):*
(a) $\mathbf{C}_n$ symmetry; (b) $\mathbf{D}_n$ symmetry; (c) $\mathbf{S}_{2n}$ symmetry.

Among all such constructions, Vortex knots, in the form of almost-planar circular braids, in which the knot filament spirals around the main symmetry axis in a monotone way, are particularly efficient in providing a desired level of symmetry with a minimal amount of bending energy (the integral of squared curvature along the whole knot trace); they contain no "unnecessary" tight turns. Most "efficient" in this respect are the simple, regular TorusKnots(s,t), which exhibit crossover-patterns of the type (ooo…o uuu…u)^s. But for large values of s and t, they start to look like industrial cooling pipes.

To make them look more like interesting, intricate knots, the crossover-sequence must be changed into a more varied pattern. The alternating pattern (ououou…ou)^s will result in a Turk's Head Knot [11], and there are also many other possible patterns. Once the mathematical knot type has been determined in this manner, the knot geometry can be refined by mapping the initial flat circular braid around a cylinder or around an ellipsoid, and the thin knot trace can be replaced with a sweep of a more interesting geometrical profile (Figure 13).

Vortex Knots in Bronze

Some of the described vortex knot constructions lead to well-known knots from the knot table. For instance Torus-knot(2,3) turned into an alternating braid will result in the Figure-8 Knot (Knot 4_1), exhibiting S_4 symmetry. Similarly, Torus-knot(3,4) turned into an alternating braid will result in the Chinese Button Knot (Knot 9_{40}), exhibiting D_3 symmetry. Figure 13 presents artistic bronze sculptures based on these two simple knots, which I have done many years ago. They were modeled by sweeping a crescent-shaped cross-section along the given knot curve. The basic geometry was fabricated on a Fused Deposition Modeling (FDM) machine; this is a 3D-printer that extrudes semi-liquid plastic with the consistency comparable to toothpaste. These ABS-plastic models were then converted into bronze in a classical investment-casting process by Steve Reinmuth in his Bronze Studio in Eugene, OR. It turns out that ABS as well as PLA plastic models sublimate cleanly in the kiln in which the plaster shell is fired. Steve Reinmuth also created the colorful patinas with a combination of heat (from a flame torch) and chemistry (from a spray bottle).

(a) (b)

Figure 13: *Bronze sculptures: (a) Knot 4_1 (Figure-8 Knot); (b) Knot 9_{40} (Chinese Button Knot).*

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